# Induced Circuits in Planar Graphs

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In McDiarmid, B. Reed, A. Schrijver, and B. Shepherd (Univ. of Waterloo Tech. Rep., 1990) a polynomial-time algorithm is given for the problem of finding a minimum cost circuit without chords (induced circuit) traversing two given vertices of a planar graph. The algorithm is based on the ellipsoid method. Here we give an $O(n^2)$ combinatorial algorithm to determine whether two nodes in a planar graph lie on an induced circuit. We also give a min-max relation for the problem of finding a maximum number of paths connecting two given vertices in a planar graph so that each pair of these paths forms an induced circuit. © 1994 Academic Press, Inc.

Let $G = (V, E)$ be an undirected graph without loops, and let $s, t$ be distinct nonadjacent vertices. We call two $s - t$ paths $P'$, $P''$ *separate* if there is no edge joining an internal vertex of $P'$ and an internal vertex of $P''$. We consider the problem of finding a maximum number of pairwise separate $s - t$ paths. For general graphs this is an NP-hard problem; this follows from Fellows [1] in which it is shown that it is NP-complete to decide if there exists an induced circuit containing $s$ and $t$.

169

We show that the problem can be solved in polynomial time for planar graphs. Moreover, we give a good characterization, based on the following concepts. Assume that $G$ is embedded in the two-sphere $S_2$. Let $C$ be a closed curve in $S_2$, not traversing $s$ or $t$. The *winding number* $w(C)$ of $C$ is, roughly speaking, the number of times that $C$ separates $s$ and $t$. More precisely, consider any curve $P$ from $s$ to $t$, crossing $C$ only a finite number of times. Let $\lambda$ be the number of times $C$ crosses $P$ from left to right, and let $\rho$ be the number of times $C$ crosses $P$ from right to left (fixing some orientation of $C$, and orienting $P$ from $s$ to $t$). Then $w(C) = |\lambda - \rho|$. (This number can be seen to be independent of the choice of $P$.)

We call a closed curve $C$ *alternate* if $C$ does not traverse $s$ or $t$ and there exists a sequence

$$(F_0, w_1, F_1, w_2, F_2, ..., w_l, F_l) \tag{1}$$

(where $l \geqslant 0$) such that

(i)   $F_0, ..., F_l$ are faces of $G$, with $F_0 = F_l$;

(ii)  $w_i$ is a vertex or edge of $G$ ($i = 1, ..., m$);

(iii) $C$ traverses vertices, edges, and faces of $G$ in the order (1).

Here, by definition, $C$ *traverses* an edge $e$ if $C$ follows $e$ from one end vertex to the other.

Let $l(C)$ denote the number $l$ in (1). Now

THEOREM A.   *Let* $G = (V, E)$ *be a graph embedded in the two-sphere* $S_2$ *and let* $s, t$ *be distinct nonadjacent vertices.*

(i)   *There exist* $k$ *pairwise separate* $s - t$ *paths if and only if* $l(C) \geqslant k \cdot w(C)$ *for each alternate closed curve* $C$.

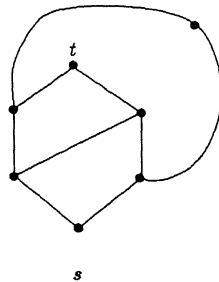(ii)  *The curves* $C$ *in* (i) *can be restricted to those with* $l(C) < |V|$ *and*



FIGURE 1

*whose intersection with G is contained in a subgraph with maximum degree two (i.e., no three $w_j$'s of (1) are mutually incident).*

(iii) *There is an $O(|V|^2)$ algorithm which finds a maximum set of pairwise separate $s - t$ paths or an appropriate alternate curve.*

Before proving the theorem, let us give a small example of a graph where an alternate curve with winding number at least two must be used. Note that a proof of nonexistence of an induced circuit containing $s, t$, by means of an alternate curve with winding number one, is equivalent to there being a vertex cut set which is a clique of size at most two. It is easily verified that the graph of Fig. 1 does not contain any induced $s, t$ circuit and yet neither does it have such a clique cut set. Other examples can be constructed for $k > 2$ (see [4]).

*Proof of Theorem A.* I. *Necessity in* (i). Let $P_1, ..., P_k$ be pairwise separate $s - t$ paths, and let $C$ be an alternate closed curve. Then $C$ intersects each $P_i$ at least $w(C)$ times. It is not hard to see that for each $i$, at least $w(C)$ of the $w_j$ in (1) are incident to a vertex in $P_i$ (defining two vertices $v'$, $v''$ to be *incident* if $v' = v''$). Since distinct $P_i$ and $P_{i'}$ are separate, there should be at least $k \cdot w(C)$ $w_j$'s, i.e., $l(C) \geqslant k \cdot w(C)$.

II. *Algorithm.* We next describe an algorithm finding for any $k$, either $k$ pairwise separate $s - t$ paths or an alternate closed curve $C$ with $l(C) < k \cdot w(C)$.

First we introduce some notation and terminology. Any $s - t$ path will be oriented from $s$ to $t$. Let $O$ be an open disk whose boundary contains $s$ and $t$. An edge $e$ (of $G$) contained in the closure $\bar{O}$ of $O$, connecting two points on the boundary of $O$, is called a *belt* relative to $O$, if any curve from $s$ to $t$ contained in $O$, must cross $e$. Let $P'$, $P''$ be two edge-disjoint $s - t$ paths, without crossings. Then $R(P', P'')$ denotes the region encircled by the closed curve $P' \cdot (P'')^{-1}$ in clockwise orientation. We call the pair $(P', P'')$ *internally separate* if $R(P', P'')$ is an open disk not containing a belt. Note that even if $(P', P'')$ is internally separate, $P'$ and $P''$ can have a vertex $v \neq s, t$ in common. Note, moreover, that $P'$ and $P''$ are separate if and only if both $(P', P'')$ and $(P'', P')$ are internally separate.

For $k = 1$ the algorithm is trivial: either there exists an $s - t$ path, or there exists a closed curve $C$ not intersecting $G$ with $w(C) = 1$ (implying $l(C) = 0 < 1 \cdot w(C)$).

Suppose now that $k > 1$, and that we have found $k - 1$ pairwise separate $s - t$ paths $P_1, ..., P_{k-1}$. In the case that $k = 2$ we assume that there exist two internally disjoint $s - t$ paths $P, Q$. If no such pair exists, then it is easy to find an appropriate alternate curve with the help of Menger's theorem. For $k = 2$ we may furthermore choose $P_1$ to be $P$.

Without loss of generality the first edges of $P_1, ..., P_{k-1}$ occur in this order clockwise at $s$. Let $P_k$ be a path "parallel" to the left of $P_1$. That is, we add to each edge traversed by $P_1$ a parallel edge at the left-hand side (with respect to the orientation of $P_1$), and $P_k$ follows these new edges. (Note that adding parallel edges does not change our problem and in the case $k = 2$ we have chosen $P_1$ so that $(P_1, P_2)$ is internally separate.) Then the first edges of $P_1, ..., P_k$ occur in this order clockwise at $s$, and each pair $(P_{i-1}, P_i)$ is internally separate $(i = 2, ..., k)$.

Now for $n = k$, $k + 1$, $k + 2$, ... we do the following. We have pairwise edge-disjoint $s - t$ paths $P_{n-k+1}, ..., P_n$, without crossings, so that the first edges of $P_{n-k+1}, ..., P_n$ occur in this order clockwise at $s$, and each pair $(P_{i-1}, P_i)$ is internally separate $(i = n - k + 2, ..., n)$.

If also the pair $(P_n, P_{n-k+1})$ is internally separate, then $P_{n-k+1}, ..., P_n$ are pairwise separate, and hence we have $k$ pairwise separate $s - t$ paths as required. If $(P_n, P_{n-k+1})$ is not internally separate, let $P_{n+1}$ be the path in $\bar{R}(P_{n-k+1}, P_{n-k+2})$ such that $(P_n, P_{n+1})$ is internally separate and such that $R(P_{n+1}, P_{n-k+2})$ is as large as possible. If $P_{n+1}$ uses an edge in $P_{n-k+2}$, then as with $P_k$, we let $P_{n+1}$ use a new parallel edge to the left. Then reset $n := n + 1$, and repeat.

III. *Correctness and running time.* Suppose we do $|V|$ iterations and let $m := k + |V|$. Consider the surface $U$ obtained in the following way. First, we cut out holes in $S_2$ at $s$ and $t$. This transforms the sphere into a cylinder where the boundaries or holes at the ends are identified with $s$ and $t$, respectively. Now make a cut from one end of the cylinder to the other to obtain a rectangle. We then obtain $U$ by taking an infinite number of copies of this rectangle and glueing them together to form an infinitely long strip whose two boundaries are again identified with the nodes $s$ and $t$. What we have described is a special instance of what is called the *universal covering surface* of some fixed surface (see [5]). In our situation $U$ is the universal covering surface of $S_2 \setminus \{s, t\}$.

Note that there are now many copies on $U$ of each point of $S_2$. Denote by $\pi$ the projection mapping $\pi \colon U \to S_2 \setminus \{s, t\}$ which maps a point of $U$ back to its associated point on $S_2$. Thus $\pi^{-1}$ maps each point of $S_2$ to an infinite set and so $\pi^{-1}[G \setminus \{s, t\}]$ is an infinite graph on $U$.

For any simple $s - t$ path $P$ in $G$, a *lifting* of $P$ is any copy of $P$ in $\pi^{-1}(P)$. If $Q$ is a lifting of $P$, we denote by $Q^1$ the lifting next to the right of $Q$. That is, $Q^1$ is to the right of $Q$ (with respect to the lifted orientation of $P$ from $s$ to $t$), and there is no other lifting of $P$ between $Q$ and $Q^1$.

By our construction, there exist liftings $Q_1, ..., Q_m$ of $P_1, ..., P_m$, respectively, so that $Q_n$ is to the right of $Q_{n-1}$ (possibly touching) for $n = 2, ..., m$, and such that $Q_{n-k+2}, ..., Q_n$ are contained in the region enclosed by $Q_{n-k+1}$ and $Q^1_{n-k+1}$ for $n = k$, $k + 1$, ..., $m$.

For each $n = k + 1, ..., m$, let $V_n$ denote the set of internal vertices of $Q_n$ which are not vertices of $Q^1_{n-k}$. Let $V_k$ be the internal vertices of $Q_k$. Since we did keep shifting, each $V_n \neq \emptyset$. Note that for any $v \in V_n$, there is an internal vertex $v'$ of $Q_{n-1}$ and a curve $C_v$ from $v'$ to $v$ such that either (a) $C_v$ traverses a face which contains $v'$ and $v$ or (b) $C_v$ traverses an edge $v'v''$ and then traverses a face containing $v''$ and $v$. For the curve $C_v$, we call $v'$ its *starting* vertex and $v$ its *end* vertex. In the case (b), the vertex $v''$ is a *middle* vertex. A vertex $v$ is *active* in some iteration if $\pi(v)$ has a lifting which is an internal vertex of one of the current paths on this iteration. Otherwise it is called *inactive*. Otherwise it is called *inactive*. Note that if a vertex $z$ is a middle vertex on iteration $i$, then it becomes inactive in iteration $i + 1$ and will only become active again on an iteration when some lifting of $\pi(z)$ occurs as an end vertex.

We claim next that $v' \in V_{n-1}$. If this is not the case, then the internal vertex $v'$ of $Q^1_{n-k-1}$ is either a vertex of $Q^1_{n-k}$ or is adjacent to an internal vertex of $Q^1_{n-k}$. This contradicts the fact that $(P_{n-k-1}, P_{n-k})$ is internally separate when $n - 1 > k$.

We now show how to construct an alternate curve. Choose $v_m \in V_m$ and for each $n = m - 1, m - 2, ..., k$, let $v_n$ be the starting vertex of $C_{v_{n+1}}$. Since $m = k + |V|$, there exist $n', n''$ with $m \geqslant n'' > n' \geqslant k$ such that $\pi(v_{n''}) = \pi(v_{n'})$. Let $D$ be the curve

$$C_{v_{n'+1}} \cdot C_{v_{n'+2}} \cdot \cdots \cdot C_{v_{n''}}, \tag{2}$$

and let $C$ be the projection $\pi \circ D$ of $D$ to $S_2$. So $C$ is an alternate closed curve with $l(C) = n'' - n'$. Next we show that $k \cdot w(C) > n'' - n'$, proving sufficiency in (i).

For any lifting $Q$ of any simple $s - t$ path $P$ and any $i \geqslant 0$, let $Q^{(i)}$ be the $i$th lifting to the right of $Q$. That is, $Q^{(0)} = Q$ and $Q^{(i+1)} = (Q^{(i)})^1$.

Let $u := \lfloor (n'' - n')/k \rfloor$. We must show $w(C) > u$. If $u = 0$, then $w(C) > u = 0$ since $v_{n''} \neq v_{n'}$. If $u > 0$, then $v_{n''}$ is strictly to the right of $Q^1_{n''-k}$ and $Q^1_{n''-k}$ is to the right of $Q^{(u)}_{n'}$ (since $Q_{n''-k}$ is to the right of $Q^{(u-1)}_{n'}$, as $n'' - k \geqslant n' + (u-1)k$). So $v_{n''}$ is strictly to the right of $Q^{(u)}_{n'}$. Therefore, $w(C) > u$.

We now analyze the running time of the algorithm and show that (ii) holds. For each $i = 1, 2, ...$ the algorithm applies a shifting procedure. This procedure either returns $i$ separate paths or an alternative curve. Each step of the shifting procedure is characterized by a path $Q_{n-1}$ which is "too" close to a previous path. More precisely, there are internal vertices $I$ of $Q_{n-1}$ which are adjacent to or contained in the set of internal vertices of $Q_{n-i}$. Hence we must shift from the vertices of $I$. It is easily checked (and standard) that the time needed to shift from a single vertex is

proportional to the size of the faces incident to the vertex. Thus since for a planar graph,

$$\sum_{F \text{ a face of } G} |V_F|$$

is $O(|V_G|)$, the shifting procedure sweeps across a single copy of $G$ (i.e., rectangle of the universal covering surface) in linear time. The point is then how many rectangles must we shift across before we find an alternate curve?

Let $K$ be the maximum number of separated paths. Then any alternate curve with length less than $|V|$ has winding number less than $|V|/K$ by necessity in (i), and so this is an upper bound on the number of rectangles. The time bound of $O(|V|^2)$ now follows. Note that after $n$ iterations we may construct a curve $C$ as in (2) which has length $n$ say and suppose that $C$ crosses $r$ liftings of $P_{i-1}$. Each time it crosses such a lifting it touches a vertex $v$ contained in a partial curve in which it is either a start, middle, or end vertex. But as we have seen, if two vertices $v'$, $v''$ occur respectively as start and end vertices on $C$ such that $\pi(v') = \pi(v'')$ and $v'$ is to the left of $v''$, then the portion of $C$ between $v'$ and $v''$ determines an alternate curve. This implies that if we have not found an alternate curve after $r$ crossings, then each internal vertex of $P_{i-1}$ has at most one lifting which occurs as a start vertex of $C$ and similarly at most one occurs as an end vertex. In fact it is straightforward to see that there is at most one lifting which occurs as a middle vertex; this can be seen by considering the curve $C$ being built in the other direction. This now easily implies (ii).

The algorithm given in the proof of the theorem can be extended for any fixed surface $S$ and any fixed $k$, to find $k$ pairwise separate $s - t$ paths in any graph embedded on $S$. It can also be shown [4] that the problem of finding a minimum-weight induced circuit traversing two given vertices $s$ and $t$ in a planar graph, is solvable in polynomial time. Moreover, finding a set of $k$ pairwise separate $s - t$ paths of minimum total weight, is solvable in polynomial time for planar graphs.

The proof of the theorem can also be extended to solve a directed version of the problem for planar diagraphs $D = (V, A)$. A collection of $s - t$ dipaths is pairwise separate if there is no arc connecting internal vertices of distinct dipaths in the collection. We call a closed curve $C$ (with clockwise orientation relative to $s$) di-alternate if $C$ does not traverse $s$ or $t$, and there exist a sequence

$$(C_0, a_1, C_1, a_2, C_2, ..., a_l, C_l) \tag{3}$$

such that

    (i)   $a_i$ is an arc of $D \backslash \{s, t\}$ with endpoints $s_i, t_i$ $(i = 1, ..., l)$;

alternate curve
projected onto $S_2$

FIGURE 2
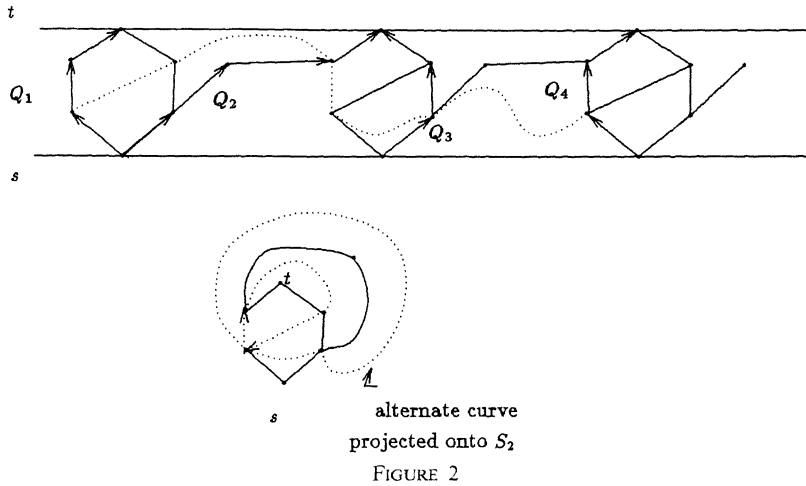
(ii)   $C_i$ is a (noncrossing) curve of positive length from $t_{i-1}$ to $s_i$ and these are the only vertices of $D$ that $C_i$ intersects ($i = 1, ..., l$ and $C_0 = C_l$);

(iii)   $C$ traverses the paths and curves given in (3) in the described order;

(iv)   each $C_i$ may cross arcs only from right to left (relative to the orientation derived from $C$) and may not cross any $a_i$.

Informally, condition (iv) requires that any arcs crossed by $C_i$ must be directed towards $s$.

THEOREM B.   *For a plane digraph $D = (V, A)$:*

(i)   *There exist $k$ pairwise separate $s - t$ dipaths if and only if $l(C) \geqslant k \cdot w(C)$ for each di-alternate closed curve $C$.*

(ii)   *A maximum number of pairwise separate $s - t$ dipaths can be found in polynomial time.*

(iii)   *The curves $C$ in (i) can be restricted to those with $l(C) < |V|$.*

We note that in the directed case we do not require the paths in the collection to be induced, i.e., they may have backwards arcs. In fact, Fellows *et al.* [2] have shown that the problem of determining whether there is a single induced $s - t$ dipath in a planar digraph is NP-complete.

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